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The stability of linear potential gyroscopic systems☆

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Abstract

The stability of linear potential systems with a degenerate matrix of gyroscopic forces is investigated. Particular attention is devoted to the case of three degrees of freedom. In a development of existing results [Kozlov VV. Gyroscopic stabilization and parametric resonance. *Prikl. Mat. Mekh.* 2001; **65**(5): 739–745], the sufficient conditions for gyroscopic stability are obtained. An algorithm for applying these conditions is proposed using the example of the problem of the motion of two mutually gravitating bodies, each of them being modelled by two equal point masses, connected by weightless inextensible rods.

We will consider the problem of the stability of the equilibrium of a linear dynamical system with n degrees of freedom, described by the equations

$$\ddot{x} + \Gamma\dot{x} + Px = 0, \quad x \in \mathbb{R}^n, \quad \Gamma^T = -\Gamma, \quad P^T = P, \quad \det P \neq 0 \tag{1}$$

The matrix P can be assumed to be a diagonal matrix.

There are many publications on the stability of the equilibrium position for the case when the potential energy has a maximum and $\det\Gamma \neq 0$ (see the brief review in Ref. 2). For commuting matrices $P\Gamma = \Gamma P$ the condition for stability is that the matrix $P - \Gamma^2/4$ should be positive definite.[3] The conditions for the solution $x = 0$ to be stable for a negative definite matrix of the potential energy P and a non-degenerate matrix of the gyroscopic forces Γ were established in Refs. 4–6, and estimates of the values of the parameters for which stability of the equilibrium position occurs were also obtained. In this case the number of degrees of freedom was necessarily even. For an odd number of degrees of freedom the matrix Γ is degenerate, and hence the results mentioned above are inapplicable. A criterion of the stability of the equilibrium of a charged particle in an electromagnetic field, and also of a general gyroscopic system with three degrees of freedom were obtained in Ref. 7. The sufficient conditions for gyroscopic stability for non-zero matrices Γ of minimum rank equal to two were obtained in Ref. 1 using the theory of parametric resonance. The purpose of the present paper is to investigate the conditions for real systems to be stable when the kinetic energy matrix is not unique.

Consider small oscillations of a system around the equilibrium position, which satisfy the linear equation with n degrees of freedom

$$M\ddot{z} + G\dot{z} + Kz = 0, \quad z \in \mathbb{R}^n \tag{2}$$

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The kinetic energy matrix of the system $M = M^T$ is positive definite, $K = K^T$ is the potential energy matrix and $G^T = -G$ is the gyroscopic forces matrix, rank $G = 2k$.

Theorem. *The sufficient conditions of gyroscopic stability of the trivial solution of Eq. (2) are that the matrices $\tilde{A}$ and $\tilde{B}$ should be positive definite, where*

$$\tilde{A} = K - GM^{-1}G/4, \quad \tilde{B} = (GM^{-1}G + \gamma^2 M)/4 - K; \quad \gamma^2 = -\lambda^2 \tag{3}$$

γ is the intensity characteristic of the gyroscopic forces and λ is the non-zero root of the characteristic equation

$$|G - \lambda M| = 0 \tag{4}$$

Remark. The theorem holds for the case when the gyroscopic forces matrix G has the form indicated below.

Proof of the Theorem. Following the well-known approach in Ref. 1, we will consider small oscillations of a dynamical system around the equilibrium position, which satisfy linear equation (1). The point $x = 0$ is the equilibrium position. Suppose the gyroscopic forces matrix has the form

$$\Gamma = \gamma S^T I_k S$$
$$I_k = \operatorname{diag}(I, \ldots, I, 0), \quad I = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \operatorname{rank} I_k = 2k, \quad \gamma > 0 \tag{5}$$

where S is an orthogonal $n \times n$ matrix. Clearly rank $\Gamma = 2k$. It can be shown that when $n = 3$ any skew-symmetric matrix of rank 2 has the form (5) where, of course, $k = 1$.

It was proved in Ref. 1 that if the matrix of the gyroscopic forces has the form (5) and $x = 0$ is the strict minimum of the changed potential energy

$$W(x) = (Px, x) + (\Gamma x, \Gamma x)/4 \tag{6}$$

and also the strict maximum of the difference $W(x) - \gamma^2(x, x)/4$, then $x = 0$ is a stable equilibrium position of system (1). Hence, the problem reduces to verifying that the matrices

$$A = (P - \Gamma^2/4), \quad B = -(P - \Gamma^2/4 - \gamma^2 E/4) \tag{7}$$

are positive definite, where E is the $n \times n$ identity matrix.

We obtain similar conditions for the equations of motion of system (2). For any matrices $M = M^T \geq 0$ and $K = K^T$, a non-degenerate matrix C exists such that

$$M = C^T C, \quad K = C^T P C$$

System (1) reduces to system (2) if we put $x = Cz$ and multiply the left-and right-hand sides of Eq. (1) on the left by C^T. The matrix Γ then takes the form $G = C^T \Gamma C$, where rank G = rank Γ, by virtue of the fact that the matrix C is non-degenerate. Then

$$W(Cz) = (PCz, Cz) + (\Gamma Cz, \Gamma Cz)/4 =$$
$$= (Kz, z) + (G^T M^{-1} Gz, z) = W(z) = (Kz, z) - (GM^{-1}Gz, z)$$

Thus, it is necessary to verify that the matrix $\tilde{A}$, defined by the first equality of (3), is positive definite.

We will show that the matrix B has the form of the matrix $\tilde{B}$, i.e. it is necessary to check the condition $\tilde{B} \geq 0$.

In fact,

$$W(x) - \gamma^2(x, x)/4 = (Kz, z) - (GM^{-1}Gz, z) + \lambda^2(Mz, z)/4$$

where $\lambda^2 = -\gamma^2$ is found from the condition

$$|\Gamma - \lambda E| = 0 \tag{8}$$

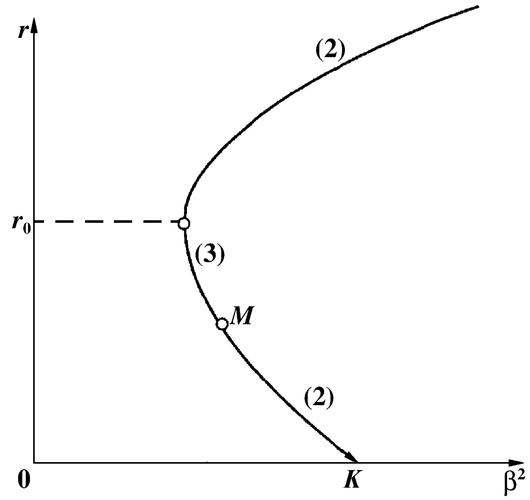

Fig. 1.

which, when $n = 3$, is equivalent to the condition

$$|\gamma S^T I S - \lambda E| = |\gamma I - \lambda S E S^T| = -\lambda(\lambda^2 + \gamma^2) = 0$$

In the general case, condition (8) takes the form

$$|C^T \Gamma C - \lambda C^T C| = 0$$

which is equivalent to Eq. (4). The theorem is proved. □

Example. Consider the plane motion of two mutually gravitating bodies with masses m_1 and m_2, each of them being modelled by two equal point masses, connected by weightless inextensible rods of lengths a_1 and a_2 respectively. The motion is defined by the coordinates r, φ, φ_1, φ_2, where r is the distance between the centres of mass of the bodies, φ is the angle between the straight line connecting the centres of mass and a fixed straight line in the plane of motion, and φ_1 and φ_2 are the angles between the rods and straight line connecting the centres of mass. Steady motions of the form $r = \text{const}$, $\dot{\varphi} = \text{const}$, $\varphi_1 = \text{const}$ and $\varphi_2 = \text{const}$ for the cases (1) $\varphi_1 = \varphi_2 = 0$, (2) $\varphi_1 = \varphi_2 = \pi/2$, (3) $\varphi_1 = 0$, $\varphi_2 = \pi/2$, (4) $\varphi_1 = \pi/2$, $\varphi_2 = 0$ were considered in Ref. 8.

The problem of the stability of the steady motion in case 1 has been investigated completely. We will consider case 2 (cases 3 and 4 can be considered similarly).

The steady motions $\varphi_1 = \varphi_2 = \pi/2$, $r = r(\beta^2)$, where β is a constant of the cyclic integral corresponding to the cyclic coordinate φ, are always Lyapunov unstable. Gyroscopic stabilization is only possible for the branch MK (see the Fig. 1), where the function $r(\beta^2)$ decreases and the degree of instability is equal to 2.[8,9] There is a change in stability at the point M, since branching of the solution occurs.

We will verify the conditions of the theorem for specific values of the parameters, corresponding to the branch MK.

The motion of the system is described by Eq. (2), where

$$z = \begin{Vmatrix} \varphi_1 \\ \varphi_2 \\ r \end{Vmatrix}, \quad M = \begin{Vmatrix} \kappa\mu_1 & -\kappa a_1^2 a_2^2 & 0 \\ -\kappa a_1^2 a_2^2 & \kappa\mu_2 & 0 \\ 0 & 0 & m \end{Vmatrix}, \quad G = \begin{Vmatrix} 0 & 0 & -\nu_1 \\ 0 & 0 & -\nu_2 \\ \nu_1 & \nu_2 & 0 \end{Vmatrix}$$

$$\kappa = \frac{m_1 m_2}{J}, \quad m = \frac{m_1 m_2}{m_1 + m_2}, \quad J = mr^2 + m_1 a_1^2 + m_2 a_2^2$$

$$\mu_j = \frac{m_j}{m_1 + m_2} a_j^2 r^2 + a_1^2 a_2^2, \quad \nu_j = \frac{2\beta m m_j}{J^2} a_j^2 r, \quad j = 1, 2$$

For the steady motion $\varphi_1 = \varphi_2 = \pi/2$ considered, the matrix K can be written in the form

$$K = f\frac{m_1 m_2}{2}\begin{Vmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{Vmatrix}$$

$$k_{jj} = a_1 a_2 b_1^- - 3r^2 b_2^+ a_j^2, \quad j = 1, 2; \quad k_{12} = k_{21} = -a_1 a_2 (b_1^- - 3r^2 b_2^-)$$

$$k_{33} = r^2 (4b_1^+ (r^2 + a_1^2 (m_1 + m_2)/m_2 + a_2^2 (m_1 + m_2)/m_1)^{-1} - 3b_2^+)$$

$$b_j^\pm = (r^2 + (a_1 - a_2)^2)^{-j-1/2} \pm (r^2 + (a_1 + a_2)^2)^{-j-1/2}, \quad j = 1, 2$$

where $f = 6.67 \times 10^{-7}\ \mathrm{km}^3/(\mathrm{t}\cdot\mathrm{s}^2)$ is the gravitational constant. We will take values of the parameters of the bodies corresponding to twin asteroids.[10]: $m_1 = 10^{18}$ t, $m_2 = 10^{15}$ t, $a_1 = 100$ km and $a_2 = 10$ km.

For $r = 100$ km all three eigenvalues of the matrix K are negative, i.e. the degree of instability is equal to three, and gyroscopic stabilization is impossible. When r is reduced, for example, to $r = 10$ km, $\beta = 0.835 \times 10^{25}$; two eigenvalues of the matrix K are negative, while one is positive, i.e. the degree of instability is equal to two; the system lies on the branch MK (see the Fig. 1). The eigenvalues of the matrix $\tilde{A}$ are: -0.33×10^{24}, 0.13×10^{22} and 0.13×10^{25}, while those of the matrix $\tilde{B}$ are: 0.5×10^{24}, -0.1×10^{25} and 0.9×10^{20}. Consequently, the matrices $\tilde{A}$ and $\tilde{B}$ are not positive definite, and hence, steady motions are gyroscopically unstable. A similar situation remains when r is reduced further.

We will now consider values of the parameters that correspond closely to actually existing objects. The twin asteroids Ida and Dactyl, discovered in 2001, have masses of $\approx 10^{14}$ and 10^{11} T, dimensions of $\approx$60 km and 1.4 km, and are a distance of $r \approx 80$ km from one another.

The degree of instability is equal to two, and the necessary condition for gyroscopic stability is satisfied. The eigenvalues of the matrix $\tilde{A}$ are: 0.12×10^{14}, 0.07×10^{22} and 0.17×10^{26}, while those of the matrix $\tilde{B}$ are: 0.92×10^{15}, 0.53×10^{22} and 0.14×10^{12}, i.e. the matrices $\tilde{A}$ and $\tilde{B}$ are positive definite, and hence gyroscopic stability in the first approximation follows.

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